Simulating Jupiter’s weather layer. Part I: Jet spin-up in a dry atmosphere

Supplementary Information

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S1. Additional figures

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Figure S1: All the terms in the Eulerian mean equation (main text Eq. 9) during Run B in equilibrium at midlatitudes, averaged over 132900–133000 d. Also shown are the adjusted Coriolis acceleration, zonal mean velocity components, adjusted zonal mean meridional velocity (main text Eq. 10), spin-down time, and latitudinal profiles at three pressure levels. In (i), $\delta_i u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$, similarly for $\delta_j$, and $[\cdots]^4$ means to apply this operator four times (Adcroft et al., 2018, Sect. 2.20).
Figure S2: Meridional structure of high latitude jets for Run A in equilibrium. As Fig. 12 in the main text but for Run A between 80–55° S, averaged over days 154760–154860. Note that the Coriolis acceleration and the adjusted Coriolis acceleration are identical, so the mass streamfunction uses the normal zonal mean meridional velocity. Eastward jets peak around 69° S and 56° S.
Figure S3: Like Fig. 13 in the main text, but showing Hovmöller diagrams for the meridional eddy momentum flux convergence at 1 bar over the whole of Runs A and B. The vertical dashed line shows when the resolution was doubled from L to M, and numbers at the ends of the colour bars show the minima and maxima.
Figure S4: As Fig. 19 in the main text but as well as showing Hovmöller plots of zonal velocity at the equator, it shows the same at two latitudes either side during Runs A and B in equilibrium at $p = 1$ bar. The three latitudes are the same as the horizontal lines in Fig. 18 in the main text.
Figure S5: Complete sequence of latitude-pressure cross sections over 1500 d during the spin-up phase of Run B at resolution M. Frames are 100 d apart, beginning at 104100 d and ending at 105500 d. Colours show angular momentum flux convergence $-\partial(\langle u'v' \rangle \cos^2 \phi)/(a \cos^2 \phi \partial \phi)$, line contours show zonal mean zonal velocity $\langle u \rangle$ (eastward flow is solid, westward flow is dashed, and zero is thicker than the other contours), and hatched regions show where the flow is barotropically unstable, i.e. where $\partial^2(u)/\partial y^2 > \beta$. 
Figure S6: As Fig. S1 but for Run B in the equatorial region in equilibrium, averaged over 132900–133000 d.
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(q) Terms at 3.98 bar.
(r) Terms at 0.40 bar.
(s) Terms at 0.04 bar.

Figure S9: As Fig. S1 but for Run A in the equatorial region during spin-up, averaged over 1300–1500 d.
Figure S10: Co-spectra and associated latitudinal profiles for Run B in equilibrium (averaged over 132800–133000 d; note this is a slightly longer average than other figures at this time, to improve the signal to noise ratio, which is quite low for this diagnostic). Contours show angular momentum flux and heat flux co-spectra. Profiles show latitudinal profiles of AM flux $\langle u'v' \rangle \cos \phi$ and heat flux $\langle v'\theta' \rangle$, which equal the sum of the co-spectral coefficients at each latitude.
Figure S11: As Fig. S10 but showing eddy AM and heat flux co-spectra and profiles at 0.4 bar during Run A during spin-up (right at the start of the run, averaged over 1300–1500 d), and in equilibrium (averaged over the final 1000 d, between 153860–154860 d).
S2. Terms in the Eulerian mean equation calculated using the MITgcm’s vector invariant discretisation

The Eulerian mean equation is (e.g. Andrews et al., 1983, Eq. 2.5)

\[ \frac{\partial (u)}{\partial t} = - \frac{\partial (u \cos \phi)}{\partial \phi} \langle v \rangle - \frac{\partial (u' v') \cos^2 \phi)}{\partial \phi} \langle \omega \rangle - \frac{\partial (u' u')}{\partial p} + f(v) \]  

(S1)

where \( \langle \cdot \rangle \) denotes a zonal mean and \( ^{'} \) the deviation therefrom, e.g. \( u(\lambda, \phi, p) = \langle u(\phi, p) \rangle + u'(\lambda, \phi, p) \). In its vector invariant form, the various forcing terms on the right hand side of the ITGcm zonal momentum equation (Adcroft et al., 2018, Sect. 2.12) are (ibid., Sect. 2.15):

\[ G_u = G_{uv}^l + G_{uw}^l \]

The first three of these terms represent the terms on the right hand side of Eq. S1 before the zonal mean is taken. In the Jupiter model \( G_{uv}^{l-dissip} = G_{uw}^{l-dissip} = 0 \) as all dissipation parameters are zero, and \( (G_u^{l'}) = 0 \) as this term is a pure longitudinal derivative (ibid., Sect. 2.15) that sums to zero when the zonal mean is taken. \( G_{cv}^{l-	au} \) is the surface stress and so is also not relevant here. The other terms introduced into the Jupiter model discussed in the main text are not discussed here.

There is a delicate balance between these terms. If they are calculated using some naive estimate of horizontal derivatives, instead of expressions derived from the same (vector invariant) equations that the model uses, then the net acceleration is typically of the same order of magnitude as individual terms, but such a net acceleration is not reflected in how the zonal mean zonal velocity actually changes over time.

The longitudinal index is \( i \), the latitudinal index is \( j \), and the vertical index is \( k \) (counting from the bottom of the atmosphere). On the longitude-latitude Arakawa C grid (ibid., Sect. 2.11.4) all the \( \Delta \gamma \) are the same; \( \Delta \gamma^f = \Delta \gamma^d = \Delta \gamma^e = \Delta \gamma^v = \Delta \gamma \), the \( \Delta \lambda \) at the same latitude are the same, and independent of longitude: \( \Delta \lambda^x = \Delta \lambda_j^x \) and \( \Delta \lambda_y = \Delta \lambda_j^y \), and the area elements spanning the same latitudes are the same, and independent of longitude: \( A^x = A^x_j \) and \( A^e = A^e_j \).

**Linear Coriolis term.** The first term on the right hand side of Eq. S1 is (ibid., Sect. 2.15.3)

\[ G_f^l = \frac{1}{\Delta \gamma^x} f \sum \sum V^j \]

where we use the following notation (ibid., Sect. 2.1):

\[ F_i^{l} = \frac{\Phi_{i-1/2,j,k} + \Phi_{i+1/2,j,k}}{2} \quad \bar{F}_i^{l} = \frac{\Phi_{l-1/2,j,k} + \Phi_{l+1/2,j,k}}{2} \]

We omit the \( h \) terms throughout as there is no bathymetry and so they all equal one. The full discretised form of this term is

\[ G_f^l \left|_{ij} \right. = \frac{1}{\Delta \lambda_j^x} \left[ \frac{f_j^x + f_{j+1}^x}{2} \right] \frac{[\Delta \lambda^x_j v_{ij} + \Delta \lambda^x_j (v_{ij+1} + v_{ij-1})]}{4} \]

Taking the zonal mean of this equation we get

\[ f(v) = \langle G_f^l \rangle = \frac{f_j^x + f_{j+1}^x (v_j + v_{j+1}) + (v_{j+1} + v_{j-1})}{2 \Delta \lambda_j^x} \]  

(S2)

where \( \langle v \rangle \) is the zonal mean of \( v \) along a longitude circle, and \( v_{ij} = \langle v \rangle + v'_{ij} \) splits the velocity into its zonal mean and eddy parts. In this term all the eddy terms sum to zero.

**Nonlinear Coriolis term.** The second term on the right hand side of Eq. S1 is (ibid., Sect. 2.15.3)

\[ G_{uv}^{l-	au} = \frac{1}{\Delta \lambda^x} f \sum \sum V^j \]

where the relative vorticity is (ibid., Sect. 2.15.1)

\[ \delta^r \Phi = \Phi_{i+1/2,j,k} - \Phi_{i-1/2,j,k} \quad \delta^r \Phi = \Phi_{i+1/2,j,k} - \Phi_{i-1/2,j,k} \]

\[ \delta^r \Phi = \Phi_{i,k+1/2} - \Phi_{i,k-1/2} \]

\[ \delta^r \Phi = \Phi_{i,k+1/2} - \Phi_{i,k-1/2} \]
The full form of this term is

\[
G_w^{\psi} \big|_{ij} = \frac{1}{2} \left[ \frac{\Delta y(v_{ij} - v_{i-1,j-1}) - (\Delta x_i u_{ij} - \Delta x_i u_{i,j-1})}{A_{i,j-1}^c} + \frac{\Delta y(v_{i,j+1} - v_{i-1,j}) - (\Delta x_i u_{i,j+1} - \Delta x_i u_{i,j})}{A_{i,j+1}^c} \right] + \frac{\Delta x_j (v_{i-1,j} + v_{i,j}) + \Delta x_j (v_{i-1,j+1} + v_{i,j+1})}{4\Delta x_j^2}
\]

Splitting into zonal and eddy terms and taking the zonal mean, we end up with three terms:

\[
-\frac{\partial (u \psi)}{\partial y} \big|_{ij} = -\frac{1}{4} \Delta x_j^2 (v_j) + \Delta x_j (v_{i-1,j}) \left[ \frac{\Delta x_j (u_j) - \Delta x_j (u_{i-1,j})}{A_{i,j-1}^c} + \frac{\Delta x_j (u_i,j+1) - \Delta x_j (u_{i-1,j})}{A_{i,j+1}^c} \right] \tag{S3}
\]

\[
-\frac{\partial v}{\partial y} \big|_{ij} = -\frac{1}{N_{lon}} \sum_{j=1}^{N_{lon}} \frac{1}{2} \left[ \Delta x_j (u_j') - \Delta x_j ( u_{i-1,j}' ) \right] + \frac{\Delta x_j (u_{i,j+1}') - \Delta x_j (u_{i-1,j}')}{A_{i,j+1}^c} \left[ \frac{\Delta x_j^2 (v_{i-1,j} + v_{i,j}) + \Delta x_j^2 (v_{i-1,j+1} + v_{i,j+1})}{4\Delta x_j^2} \right] \tag{S4}
\]

\[
-\frac{\partial v}{\partial x} \big|_{ij} = \frac{\Delta y}{N_{lon} \Delta p_k} \sum_{j=1}^{N_{lon}} \frac{1}{2} \left[ \Delta x_j^2 (v_{i-1,j}' - v_{i-1,j}) + \Delta x_j^2 (v_{i,j+1}' - v_{i,j+1}) \right] \left[ \frac{\Delta x_j^2 (v_{i-1,j} + v_{i,j}) + \Delta x_j^2 (v_{i-1,j+1} + v_{i,j+1})}{4\Delta x_j^2} \right] \tag{S5}
\]

where \(N_{lon}\) is the number of longitudinal grid points, wrapping around in longitude where necessary. The term \(\langle \psi \partial (v) \rangle / \partial x\) is identically zero. In the continuous equations Eq. S5 also equals zero, but because of the discretisation it is actually nonzero (but very small). It has a very similar appearance to the horizontal metric term derived from the flux form equations (metric terms are explicitly excluded from the vector invariant equations).

**Shear term.** The third term on the right hand side of Eq. S1 is (ibid., Sect. 2.15.4)

\[
G_w^{\psi v} = \frac{1}{A^w \Delta r} A^w \delta_j^k u
\]

Where we have omitted the non-hydrostatic term as we do not use it. The full form of this term in pressure coordinates is

\[
G_w^{\psi v} \big|_{ik} = -\frac{(\omega_{i-1,k} + \omega_k)(u_{i,k} - u_{i,k-1}) + (\omega_{i-1,k+1} + \omega_{i+1,k})(u_{i+1,k} - u_{i,k})}{4\Delta p_k}
\]

where \(\Delta p_k^f = p_{i,k}^f - p_{i+1,k}^f\) is the pressure difference between faces \(k\) and \(k+1\), and is defined to be positive. Splitting into zonal and eddy terms and computing the zonal mean, we end up with two terms:

\[
-\frac{(\omega)}{\partial p} \bigg|_{i,k} = -\frac{(\omega)_k(u_{i,k} - (u_{i,k-1}) + (\omega)_{k+1}(u_{i,k+1} - (u_{i,k}))}{2\Delta p_k^f} \tag{S6}
\]

\[
-\frac{(\omega)}{\partial p} \bigg|_{i,k} = -\frac{1}{4N_{lon}\Delta p_k} \sum_{j=1}^{N_{lon}} \left[ (\omega_{i-1,k} + \omega_{i,k})(u_{i,k} - u_{i,k-1}) + (\omega_{i-1,k+1} + \omega_{i,k+1})(u_{i,k+1} - u_{i,k}) \right] \tag{S7}
\]

**Eddy fluxes.** At this point we have three of the five terms in the Eulerian mean equations, the first, third, and fifth terms, but we do not yet have the eddy fluxes. We require \(-\partial (u' \psi')/\partial y\) but have \(-\langle v' \partial u' / \partial y \rangle\), and \(-\partial (u' \omega')/\partial p\) but have \(-\langle \omega' \partial u' / \partial p \rangle\). These are related by

\[
-\frac{\partial (u' \psi')}{\partial y} = -\left\langle v' \frac{\partial u'}{\partial y} \right\rangle - \left\langle u' \frac{\partial \psi'}{\partial y} \right\rangle \quad \text{and} \quad -\frac{\partial (u' \omega')}{\partial p} = -\left\langle \omega' \frac{\partial u'}{\partial p} \right\rangle - \left\langle \frac{\partial \omega'}{\partial p} \right\rangle
\]

By exploiting the symmetry of these two expressions, we can draw stencils for the terms we do know and use the symmetry of these stencils about a reflection in \(i - j\) (or \(i - k\) for the vertical term) to write down the other terms immediately, without doing any derivations:

\[
-\left\langle u' \frac{\partial \psi'}{\partial y} \right\rangle = -\frac{1}{4N_{lon}\Delta p_k} \sum_{j=1}^{N_{lon}} \left[ (\Delta x_j^2 (v_{i-1,j}' - v_{i-1,j}) - \Delta x_j^2 (v_{i,j}' - v_{i,j})) + (\Delta x_j^2 (v_{i-1,j+1}' - v_{i-1,j+1}) - \Delta x_j^2 (v_{i,j+1}' - v_{i,j+1}) + (u_{i,k}' + u_{i,k+1})(\omega_{i,k+1} - \omega_{i,k}) \right] \tag{S8}
\]

\[
-\left\langle u' \frac{\partial \omega'}{\partial p} \right\rangle = -\frac{1}{4N_{lon}\Delta p_k} \sum_{j=1}^{N_{lon}} \left[ (u_{i-1,k} + u_{i,k})(\omega_{i-1,k} - \omega_{i,k}) + (u_{i,k} + u_{i,k+1})(\omega_{i+1,k} - \omega_{i,k}) \right] \tag{S9}
\]
If these two terms sum to zero then we can add both to \( \langle G_u \rangle \) and not change its sum, which then allows us to express \( \langle G_u \rangle \) in terms of the standard eddy fluxes \(-\partial\langle u'v' \rangle /\partial y\) and \(-\partial\langle u'\omega' \rangle /\partial p\). The sum of these two terms is
\[
-\left\langle u'\frac{\partial v'}{\partial y} \right\rangle - \left\langle u'\frac{\partial \omega'}{\partial p} \right\rangle = -\left\langle u' \left(\frac{\partial v'}{\partial y} + \frac{\partial \omega'}{\partial p}\right) \right\rangle = -\left\langle u' \left(\frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial p}\right) \right\rangle.
\]

By continuity,
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0
\]
taking the zonal mean,
\[
\frac{\partial (v)}{\partial y} + \frac{\partial (\omega)}{\partial p} = 0
\]
So the latter two terms sum to zero, and the first and second can be rewritten as
\[
\left\langle u'\frac{\partial u}{\partial x} \right\rangle = \left\langle u'\frac{\partial u}{\partial x} - \langle u \rangle \frac{\partial u}{\partial x} \right\rangle = \frac{1}{2} \left[ \frac{\partial (u^2)}{\partial x} \right] - \langle u \rangle \frac{\partial (u)}{\partial x} = 0
\]
So, in summary, we may rewrite the Eulerian mean equation Eq. S1 as
\[
\frac{\partial (u)}{\partial t} = \text{Eq. S3} \quad \text{Eq. S1 term 1 – Meridional mean momentum flux convergence}
+ \text{Eq. S4 + Eq. S8} \quad \text{Eq. S1 term 2 – Meridional eddy momentum flux convergence}
+ \text{Eq. S6} \quad \text{Eq. S1 term 3 – Vertical mean momentum flux convergence}
+ \text{Eq. S7 + Eq. S9} \quad \text{Eq. S1 term 4 – Vertical eddy momentum flux convergence}
+ \text{Eq. S2} \quad \text{Eq. S1 term 5 – Coriolis term}
+ \text{Eq. S5} \quad \text{Discretised version of \( \langle v'\partial v'/\partial x \rangle \)}
\]

References
